Chapter 5 of Textbook

Main Object: linear operator $T: V \longrightarrow V$

Main Goal: Diagonalization

Main Tool: Eigenvector & Invariant Subspaces.

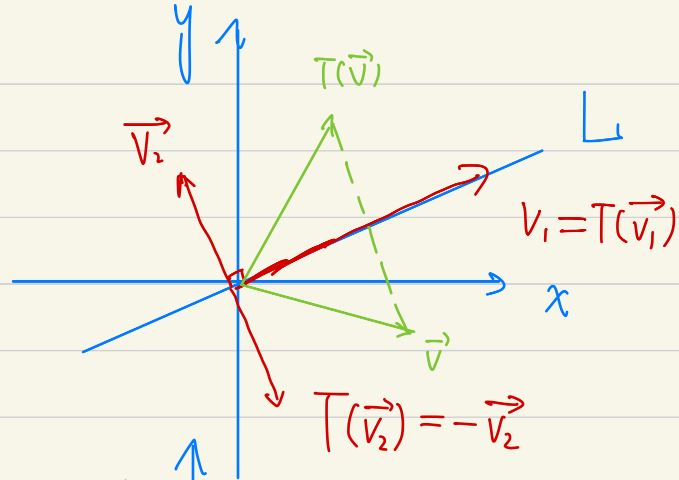
§ Eigenvalues and Eigenvectors.

Def: Let $T \in \mathcal{L}(V)$. A vector $\vec{0} \neq \vec{v} \in V$ is an eigenvector of T if $\exists\, \lambda \in F$ s.t. $T(\vec{v}) = \lambda \cdot \vec{v}$.

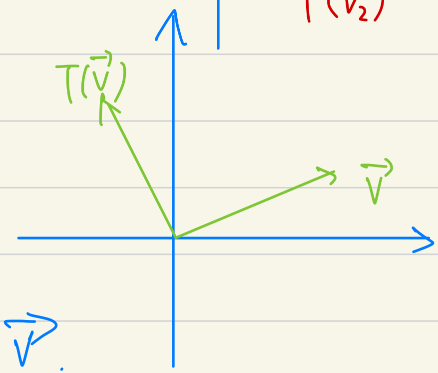
In this case, $\lambda \in F$ is called the eigenvalue asso. with the eigenvector $\vec{v}$

Ex: (1) T as geometric motion.

- Reflection.



- Rotation (by 90°) $T: \mathbb{R}^2 \to \mathbb{R}^2$



Clearly, $\forall\ \vec{v} \neq 0 \in V$.

$T(\vec{v})$ is not a multiple of $\vec{v}$.

$\Rightarrow$ T has no eigenvector, hence no eigenvalue.

(2) T as operator on function space:

e.g. $T: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ $\qquad C^\infty(\mathbb{R}) = \{\text{smooth functions}\}$

$f \rightsquigarrow T(f) = f'$

$T \in \mathcal{L}(C^\infty(\mathbb{R}))$

Solve: $T(f) = f' = \lambda f$. $\quad f \neq 0$.

$\Rightarrow f(t) = c \cdot e^{\lambda t} \quad (c \neq 0)$

Hence, Any $\lambda \in \mathbb{R}$ is an eigenvalue of T, corresponding to the eigenvector $c \cdot e^{\lambda t}$. $c \neq 0$.

(3). $T = L_A : F^n \longrightarrow F^n$. where $A \in M_{n\times n}(F)$.

$$T\vec{v} = \lambda \vec{v} \iff A\cdot\vec{v} = \lambda\cdot\vec{v}.$$

Def: eigenvalue & eigenvector of matrix A.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: $A\vec{v_1} = 3\cdot\vec{v_1}$;

$\vec{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$: $A\vec{v_2} = -\vec{v_2}$.

Q: Determine eigenvector & eigenvalue of arbitrary lin. operator T ?

A: General method uses Characteristic polynomial.

$\rightsquigarrow$ roots give eigenvalue

$\rightsquigarrow$ Solve lin. eq. to find associated eigenvectors.

Def: • Given $A \in M_{n\times n}(F)$, the characteristic polynomial is defined as

$$f_A(t) := \det(A - tI_n)$$

• Given $T \in \mathcal{L}(V)$. $\dim V = n$. β: ordered basis for V.

Define: $$f_T(t) := \det([T]_\beta - tI_n)$$

Ex: (1) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

$$f_A(t) = \det \underbrace{\begin{pmatrix} 1-t & 1 \\ 4 & 1-t \end{pmatrix}}_{= A - tI} = (1-t)^2 - 4$$

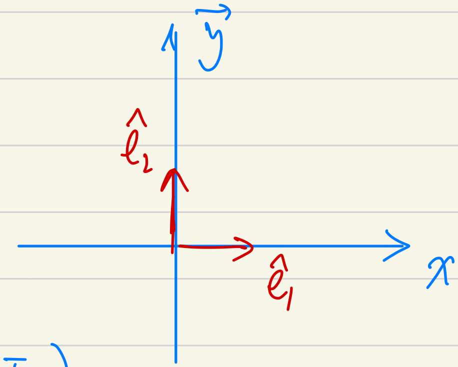
$$= t^2 - 2t - 3$$

$$= (t-3)(t+1)$$

(2). Char. poly of $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by 90°.

In the standard basis β,

then $[T]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Therefore, $f_T(t) = \det\left([T]_\beta - tI\right)$

$$= \det\begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1.$$

Prop: $f_T(t)$ is well-defined. i.e., independent of the choice of β.

Pf: If β' is another ordered basis for V, then $[T]_{\beta'} = Q^{-1} \cdot [T]_\beta \cdot Q$.

Then $$\det([T]_{\beta'} - tI_n) = \det\left(Q^{-1}([T]_\beta - tI_n)Q\right)$$
$$= \cancel{\det(Q)^{-1}} \cdot \det([T]_\beta - tI_n) \cdot \cancel{\det Q}$$
$$= \det([T]_\beta - tI_n)$$

$\square$

Prop: $f_A(t) = (-1)^n t^n + (-1)^{n-1}\operatorname{tr}A \cdot t^{n-1} + \cdots + \det A$.

(Exercise)

Prop: $A \in M_{n\times n}(F)$, λ eigenvalue

$\iff \underbrace{\det(A-\lambda I_n)}_{= f_A(\lambda)} = 0$. $\iff$ λ root of $f_A(t)$.

pf: λ eigenvalue $\iff A\cdot\vec{v} = \lambda\cdot\vec{v}$ for some $\vec{v} \neq \vec{0}$.

$\iff (A-\lambda I_n)\cdot\vec{v} = 0$

$\iff A-\lambda I_n$ is not invertible (singular)

$\iff \det(A-\lambda I_n) = 0$. $\square$

Prop: Similarly, λ eigenvalue of $T \in \mathcal{L}(V) \iff \lambda$ root of $f_T(t)$.

pf: $\quad \lambda$ eigenvalue

$\iff T\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$

$\iff (T - \lambda I_V) \cdot \vec{v} = 0$

$\iff ([T]_\beta - \lambda \cdot I_n)\, [\vec{v}]_\beta = 0$.

$\iff \det([T]_\beta - \lambda \cdot I_n) = 0$ $\quad\square$

$f_T(\lambda)$

Def: $T \in \mathcal{L}(V)$. and λ eigenvalue of T.

The subspace $\boxed{E_\lambda := N(T - \lambda I_V) = \{\vec{v} \in V : T(\vec{v}) = \lambda \vec{v}\} \subset V}$

null space

is called the eigenspace of T corresponding to λ.

In particular, for $A \in M_{n\times n}(F)$.

the eigenspace $E_\lambda = \{\vec{v} \in F^n : A \cdot \vec{v} = \lambda \cdot \vec{v}\} \subset F^n$.

Example 1: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \Rightarrow f_A(t) = (t-3)(t+1)$

So the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

For $\lambda_1 = 3$. $E_{\lambda_1} = N(A - \lambda_1 \cdot I)$

$$= N\left(\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\right)$$

$$= \left\{ c \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} : c \in \mathbb{R} \right\}$$

For $\lambda_2 = -1$. $E_{\lambda_2} = N(A - \lambda_2 I)$

$$= N\left(\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\right).$$

$$= \left\{ c \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$$x^2 \rightsquigarrow 2x+3x^2$$
$$x \rightsquigarrow 1+2x$$
$$1 \rightsquigarrow 1$$

Example 2: $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$$f(x) \rightsquigarrow Tf(x) = f(x) + (x+1)\cdot f'(x).$$

(Check: this is a linear operator)

Find eigenvector & eigenvalue of T:

Apply the general method: Take $\beta = \{1, x, x^2\}$ the standard basis for $P_2(\mathbb{R})$

Then $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

- The characteristic poly $\det([T]_\beta - tI_3)$

$$= \det \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix}$$

$$= (1-t)\cdot(2-t)\cdot(3-t)$$

$\Rightarrow$ T has eigenvalues $\lambda_1 = 1$. $\lambda_2 = 2$, $\lambda_3 = 3$.

- For $\lambda_1 = 1$.

$$E_{\lambda_1} = N([T]_\beta - \lambda_1 \cdot I)$$

$$= N\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad a \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

Obtain $\vec{v} \in V$ from coord. $[\vec{v}]_\beta$

$$\cong \{a : a \in \mathbb{R}\} \qquad \subset P_2(\mathbb{R}) \xrightarrow{\phi_\beta} \mathbb{R}^3$$

Indeed, $T(a) = a$.

· For $\lambda_2 = 2$.

$$E_{\lambda_2} = N\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ a \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

$$\cong \{ a(1+x) \} \subset P_2(\mathbb{R}).$$

Indeed, $T(1+x) = (1+x) + (x+1) = 2(1+x)$

For $\lambda_3 = 3$.

$$E_{\lambda_3} = N\begin{pmatrix}-2 & 1 & 0\\ 0 & -1 & 2\\ 0 & 0 & 0\end{pmatrix}$$

$$= \left\{ a\begin{pmatrix}1\\2\\1\end{pmatrix},\ a\in\mathbb{R}\right\} \subset \mathbb{R}^3$$

$$\cong \left\{ a(1+2x+x^2) \right\} \subset P_2(\mathbb{R}).$$

Indeed. $T(\underbrace{1+2x+x^2}_{=(1+x)^2}) = (1+x)^2 + (x+1)\cdot 2(1+x) = 3(1+x)^2$ $\square$